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Quasi-Chaotic Property of the Prime-Number Sequence

Richard L. Liboff[1] and Michael Wong[2]

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The prime-number sequence, viewed as the spectrum of eigenvalues of random matrices, is found to be quasi-chaotic. Plots of histograms of prime-number nearest-neighbor spacing Δp at various values of total number of integers indicate rough agreement with the Wigner distribution and illustrate level repulsion. A global maximum of these curves is noted at $\Delta p = 6$. Numerical work further implies that in any maximum integer sampling, no matter how large, a finite number of nearest neighbor spacings do not occur. This quasi-chaotic property of the prime-number sequence supports the conjecture that a formula for the nth prime does not exist. A rule for missing spacings is inferred according to which, as maximum number of integers $\mathcal{N}$ increases, nearest neighbor vacancies corresponding to smaller $\mathcal{N}$ vanish and new, larger value vacancies appear. In addition, early values of these histograms illustrate a rough oscillatory behavior with periodicity $\delta[\Delta p] \simeq 6$. A corollary to the results implies that zeros of the Riemann zeta function likewise comprise a quasi-chaotic sequence. Application of these findings to the resonant spectra of excited nuclei is noted.

1. INTRODUCTION

Quantum chaos of a system may be described in terms of the theory of Gaussian distributions of random matrices and the spectral statistics of the spacing of eigenvalues of these matrices (Berry and Tabor, 1977; Porter, 1965; Mehta, 1967; Reichl, 1992; Haake, 1990, Liboff and Seidman, 1993). In the formalism stemming from symmetries of the given system, three classes of matrices emerge: real symmetric, Hermitian, and real quaternion, which in turn may be diagonalized by orthogonal, unitary,

[1] Schools of Electrical Engineering and Applied Physics, Cornell University, Ithaca, New York 14853.
[2] National Semiconductor Corporation, Santa Clara, California 95052-0090.

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and symplectic similarity transformations, respectively. Corresponding distributions of nearest neighbor eigenvalues carry the acronyms GOE, GUE, and GSE. The first of these (Gaussian orthogonal ensemble) is also called the Wigner–Dyson (Wigner, 1967; Dyson, 1962) or, simply, the Wigner distribution. A histogram of nearest neighbor increments of eigenvalues which roughly resembles any of these distributions reflects a nonintegrable system, whereas a Poisson-like distribution is indicative of an integrable system. Thus, for example, nearest neighbor increments of the energy spectrum of the circular quantum billiard gives a Poisson-like histogram, whereas the energy spectrum of chaotic billiards such as the stadium or Sinai billiard give a Wigner-like histogram (McDonald and Kaufman, 1979; Bohigas *et al.*, 1984). In the present work this formalism is applied to the prime number sequence. As previously noted (e.g., Reichl, 1992, Chapter 2; Haake, 1990, Chapter 4), these spectral statistics provide a criterion for chaotic behavior. This interpretation is adopted in the present work. We term the set of eigenvalues related to a chaotic system with spectral properties such as described above *a chaotic sequence.* When applied to the prime-number sequence it is found that there is rough agreement with the Wigner distribution beyond the maximum of this distribution, but that the Wigner distribution falls off more rapidly than does the corresponding prime-number incremental histogram in the large-incremental-number domain. Thus, although spectral properties of the prime-number sequence exhibit level repulsion, this difference in the large-incremental-number interval is taken to describe a quasi-chaotic sequence. In a related study of quantum chaos of dynamical systems, analogies were made between classical periods and prime numbers, and energy levels and zeros of Riemann's zeta function (Berry, 1993).

Spectral densities of chaotic systems reflect level repulsion. Early studies of nuclear resonances in U^{238} under neutron bombardment clearly exhibit level repulsion related to the Wigner distribution (Gutzwiller, 1990, Section 16.4). Studies addressing the application of prime numbers to physics have suggested a relation between prime-number sequences and the spectra of excited nuclei (Cipra, 1996). In this context, spectral densities of resonance data of nuclei with mass numbers ranging from $A = 50$ to $A = 230$ were compiled (Michell *et al.*, 1991). It was observed that the spectral resonance of lighter nuclei exhibit GOE chaotic behavior, whereas heavier nuclei exhibit Poisson-like integrable behavior. Due to Euler's formula relating the Riemann zeta function to primes, it is concluded that zeros of these functions likewise comprise a quasi-chaotic sequence. Large-order zeros of the zeta function have also been identified with spectra of excited nuclei (Cipra, 1996).

2. ANALYSIS

The probability $P(s)\,ds$ for eigenvalue spacing in the interval $(s, s + ds)$ related to a Gaussian distribution of random 2×2 matrices is given by the distribution (Berry and Tabor, 1977; Porter, 1965; Mehta, 1967; Reichl, 1992)

$$P(s) = \frac{|s|^{\beta} e^{-s^2/8a^2}}{a^{\beta+1} 2^{(3\beta+3)/2} \Gamma((\beta + 1)/2)} \tag{1}$$

where the values $\beta = 1, 2, 4$ correspond respectively, to the GOE, GUE, and GSE cases, and a is a constant. Thus, the GOE distribution has the form

$$P_{\mathrm{O}}(s) \propto |s| \exp(-s^2/8a^2) \tag{2a}$$

the GUE distribution has the form

$$P_{\mathrm{U}}(s) \propto s^2 \exp(-s^2/8a^2) \tag{2b}$$

and the GSE distribution has the form

$$P_{\mathrm{S}}(s) \propto s^4 \exp(-s^2/8a^2) \tag{2c}$$

In the present work the sequence of prime numbers is viewed as the spectrum of eigenvalues of random 2×2 matrices. A histogram of numbers of primes $N(\Delta p)$ in spacings Δp of adjacent prime-number values was made from the first 50,000, 10^6, and 10^7 positive integers, respectively. In compiling the best fit of $P(s)$ to the given histogram, it was noted that the P_{S} and P_{U} distributions rise too slowly at $s = 0^+$, and that the Wigner distribution gives the best fit. This finding implies that the prime-number sequence relates to real, symmetric matrices.

2.1. Distributions

The nature of the numerics in the present formulation is as follows. Let $\mathcal{N}$ denote the maximum integer in any run. For $\mathcal{N} \gg 1$, it is found that the general features of the histogram remain, namely, the peak is always at $\Delta p = 6$, this maximum grows with $\mathcal{N}$, as does the maximum value of Δp, although the related value of $N(\Delta p)$ is vanishingly small. The resulting distribution corresponding to $\mathcal{N} = 50{,}000$ is given by

$$P(s) = \frac{2}{72} s e^{-s^2/72} \tag{3a}$$

$$\overline{P}(s) = \frac{300 \times 72}{2} P(s) \tag{3b}$$

where $P(s)$ is normalized and is chosen with maximum at $s = 6$, corresponding to $a = 3$ in (1). The renormalized distribution (3b) is constructed to better exhibit a rough fit with the Wigner distribution (Fig. 1). The same procedure is followed for the remaining two examples. Distributions corresponding to $\mathcal{N} = 10^6$ are

$$P(s) = \frac{2}{128} se^{-s^2/128} \tag{4a}$$

$$\bar{P}(s) = \frac{2500 \times 128}{2} P(s) \tag{4b}$$

whose maximum values were chosen at $s = 8$, corresponding to $a = 4$ (Fig. 2). Distributions corresponding to $\mathcal{N} = 10^7$ are

$$P(s) = \frac{2}{162} se^{-s^2/162} \tag{5a}$$

$$\bar{P}(s) = \frac{1.8 \times 10^4 \times 162}{2} P(s) \tag{5b}$$

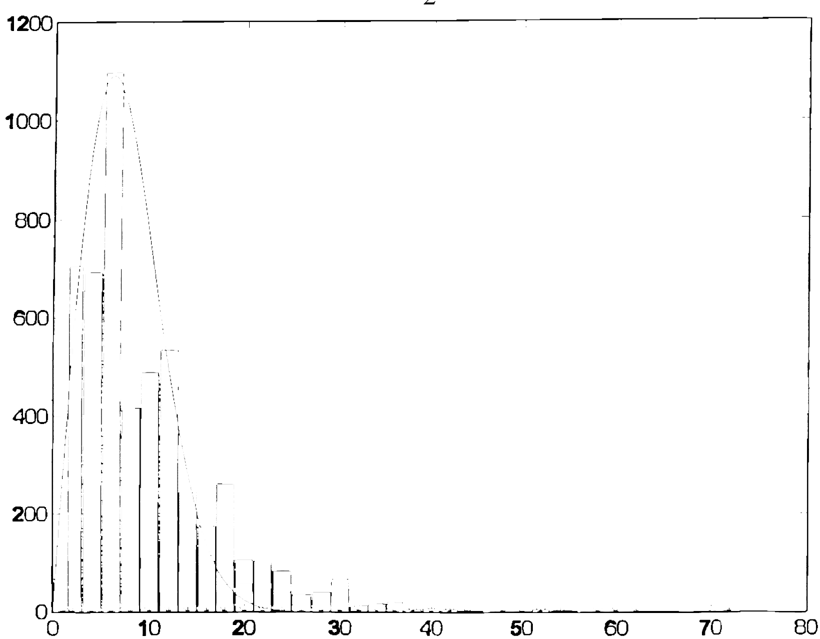

Fig. 1. Superimposed histogram of the number of primes $N(\Delta p)$ in the nearest neighbor spacing Δp for the first 50,000 integers, and corresponding renormalized Wigner distribution $\bar{P}(s)$ illustrating level repulsion and a rough fit of the histogram to the distribution. In this and the following figures, a missing star on the Δp axis indicates the absence of the related Δp value.

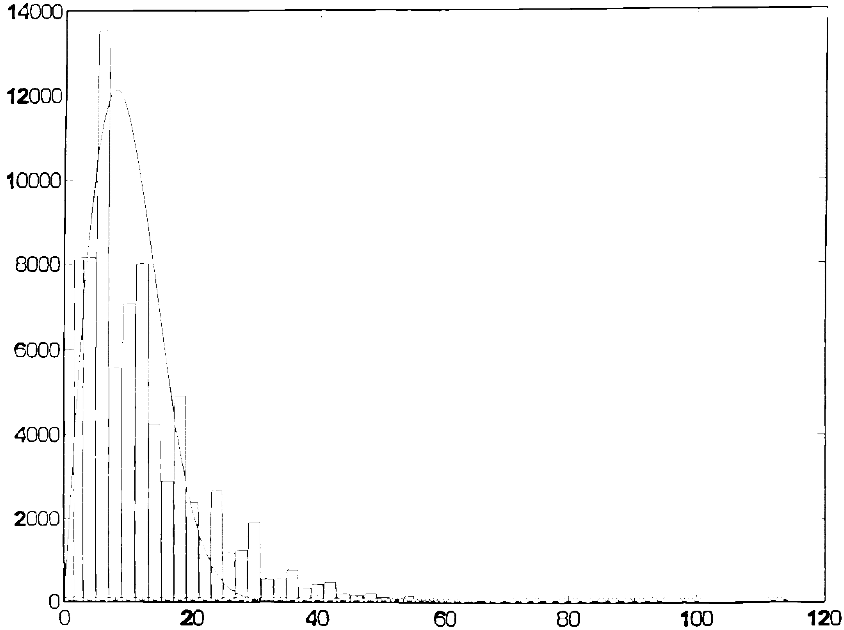

Fig. 2. Superimposed histogram of the number of primes $N(\Delta p)$ in the nearest neighbor spacing for the first 10^6 integers, and corresponding renormalized Wigner distribution $P(s)$ illustrating level repulsion and a rough fit of the histogram to the distribution.

whose maximum values were chosen at $s = 9$ corresponding to $a = 4.5$ (Fig. 3).

One notes that in all cases the $N(\Delta p)$ curves have a global maximum at $\Delta p = 6$. In these figures the related histogram includes the value $N(0) = 0$, which corresponds to the property that primes do not repeat, and the value $N(1) = 1$, which corresponds to the interval (2, 3).

In the course of this numerical study it was found that for sufficiently large total-integer number, a finite set of nearest neighbor increments are missing. Thus, for example, in the sampling of the first 50,000 integers, prime numbers with nearest neighbor spacings $\Delta p = 46, 64, 66, 68, 70$ do not occur. For this value of $\mathcal{N}$, the program gives zero $N(\Delta p)$ for $\Delta p > 72$. In the sampling of the first 10^6 integers, prime numbers with nearest neighbor spacings $\Delta p = 94, 102, 104, 106, 108, 110$ do not occur. For this value of $\mathcal{N}$, the program gives zero $N(\Delta p)$ for $\Delta p > 114$. In the sampling of the first 10^7 integers, prime numbers with nearest neighbor spacings $\Delta p = 142, 144, 150$ do not occur. For this value of $\mathcal{N}$, the program gives zero $N(\Delta p)$ for $\Delta p > 154$. Readouts strongly suggest that as the total number of integers increases, nearest neighbor vacancies corresponding to smaller $\mathcal{N}$ values vanish and new, larger value vacancies

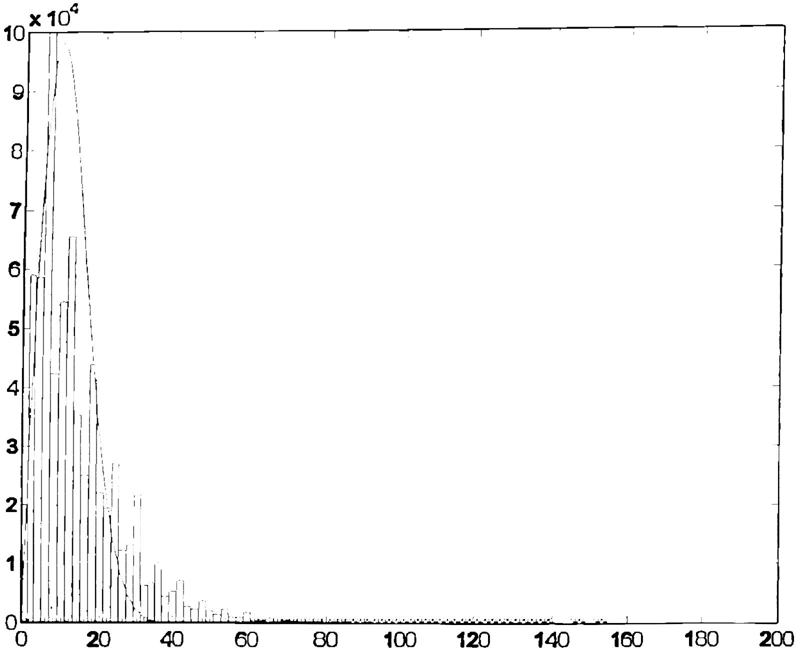

Fig. 3. Superimposed histogram of the number of primes $N(\Delta p)$ in the nearest neighbor spacing for the first 10^7 integers, and corresponding renormalized Wigner distribution $P(s)$ illustrating level repulsion and a rough fit of the histogram to the distribution. In this figure the maximum value of $N(\Delta p)$ at $\Delta p = 6$ has the value 10^5.

appear. In addition, early values of these curves illustrate a rough oscillatory behavior of $N(\Delta p)$ with periodicity $\delta[\Delta p] \simeq 6$.

2.2. Prime-Number Theorems

Here we give some well-known theorems regarding prime numbers (Niven and Zuckertman, 1990; Ore, 1948; Ellison, 1985; Kraitchik, 1992, 1926; Heath-Brown, 1988) related to the present work.

A. There are arbitrarily large gaps in the series of primes.

B. Lejeune–Dirichlet theorem: For any positive integer pair (a, b) with GCD $= 1$, there exists an infinite number of primes in the sequence

$$an + b \tag{6a}$$

where n is a positive integer. Thus, for example, there is an infinite number of primes in the respective sequences $3n + 8$, $5n + 9$, etc.

C. Euler's formula:

$$\zeta(z) = \prod_p \left(\frac{p^z}{p^z - 1} \right) \tag{6b}$$

where $\zeta(z)$ is the Riemann zeta function, and the product is over all primes.

D. Let $\Pi(p)$ denote the number of primes $\leq p$. Then Tchebychef's inequality states that

$$0.921 \frac{p}{\ln p} \leq \Pi(p) \leq 1.06 \frac{p}{\ln p} \tag{6c}$$

E. The prime-number theorem states that

$$\lim_{p\to\infty} \left(\frac{\Pi(p)}{p/\ln p} \right) = 1; \qquad \Pi(p) \sim p/\ln p \tag{6d}$$

The right asymptotic relation is valid in the limit of large p. The prime-number theorem also may be written (after Gauss)

$$\lim_{p\to\infty} \left(\frac{Li(p)}{p/\ln p} \right) = 1; \qquad Li(p) \equiv \int_2^p \frac{dy}{\ln y} \tag{6e}$$

The latter relations indicate that primes grow sparse as p increases. Thus, for example, at $p[10^7]$, $\Pi'(p) \simeq 0.062$, (where $p[x]$ denotes the prime closest to x). This prime diminishment property was demonstrated in extensive numerical studies by Kraitchik (1922, 1926). The latter observation, as well as property 1 related to gaps in the prime-number sequence, are consistent with the incremental distribution of primes discussed herein. Properties of increments of consecutive primes is discussed by Heath-Brown (1988).

2.3. Riemann Zeta Function

With (6b) we see that zeros of the Riemann zeta function are functions of the prime numbers. Consequently, with the present interpretation, one may infer that these zeros likewise comprise a quasi-chaotic sequence. This interpretation is in accord with the observation that the distribution of spacings of zeros of the zeta function conforms roughly with a GUE which, as noted above, relate to Hermitian matrices (Berry, 1988; Berry, 1986). This property is consistent with present findings in the following manner. The complex counterparts of real, symmetric matrices related to the prime-number system are Hermitian matrices related to the complex system of zeros of the Riemann zeta function. In addition, as noted above, both prime numbers and zeros of the zeta function have been associated with the spectra of excited nuclei (Cipra, 1996).

3. SUMMARY

We have viewed the sequence of prime numbers as the spectrum of eigenvalues of random 2×2 matrices. The spectral statistics of the prime-number sequence for the first 5×10^4, 10^6, and 10^7 integers were examined, respectively. In each case, a histogram of the number of prime numbers in adjacent prime-number intervals was found to be roughly Wigner-like, illustrating level repulsion and relating to real, symmetric matrices. However, it was noted that although the curves imply level repulsion, in the domain of large prime-number increments, the Wigner distribution falls off more rapidly than does the corresponding incremental prime-number histogram. Accordingly, these results were interpreted to imply a quasi-chaotic property of the prime-number sequence. In each of these readings, nearest neighbor spacing $\Delta p = 6$ was found to be a global maximum. A rule for missing spacings was inferred according to which, as the maximum number of integers $\mathcal{N}$ increases, nearest neighbor vacancies corresponding to smaller $\mathcal{N}$ vanish and new, larger value vacancies appear. In addition, early values of these curves illustrate a rough oscillatory behavior of $N(\Delta p)$ with periodicity $\delta[\Delta p] \simeq 6$. A corollary to these findings implies that zeros of the Riemann zeta function likewise comprise a quasi-chaotic sequence. The quasi-chaotic property of the prime-number sequence as well as the relative fluctuations of $N(\Delta p)$ with varying $\mathcal{N}$ support the conjecture that a general formula for the nth prime does not exist. Application of this study to the spectra of excited nuclei was noted.

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